Problem 3.16

Show that two noncommuting operators cannot have a complete set of common eigenfunctions. *Hint:* Show that *if* $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions, then $\left[\hat{P}, \hat{Q}\right] f = 0$ for any function in Hilbert space.

Solution

Suppose that $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions.

$$\hat{P} f_n(x) = p_n f_n(x)$$

$$\hat{Q} f_n(x) = q_n f_n(x)$$

Because the set of eigenfunctions is complete, any function in Hilbert space can be written as a linear combination of them.

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \tag{3.11}$$

The aim is to show that $\left[\hat{P}, \hat{Q}\right] = 0$ for any function in Hilbert space.

$$\begin{aligned}
\left[\hat{P}, \hat{Q}\right] f(x) &= \left(\hat{P}\hat{Q} - \hat{Q}\hat{P}\right) f(x) \\
&= \hat{P}\hat{Q} f(x) - \hat{Q}\hat{P} f(x) \\
&= \hat{P}\hat{Q} \sum_{n=1}^{\infty} c_n f_n(x) - \hat{Q}\hat{P} \sum_{n=1}^{\infty} c_n f_n(x) \\
&= \hat{P} \sum_{n=1}^{\infty} c_n [\hat{Q} f_n(x)] - \hat{Q} \sum_{n=1}^{\infty} c_n [\hat{P} f_n(x)] \\
&= \hat{P} \sum_{n=1}^{\infty} c_n [q_n f_n(x)] - \hat{Q} \sum_{n=1}^{\infty} c_n [p_n f_n(x)] \\
&= \sum_{n=1}^{\infty} c_n q_n [\hat{P} f_n(x)] - \sum_{n=1}^{\infty} c_n p_n [\hat{Q} f_n(x)] \\
&= \sum_{n=1}^{\infty} c_n q_n [p_n f_n(x)] - \sum_{n=1}^{\infty} c_n p_n [q_n f_n(x)] \\
&= \sum_{n=1}^{\infty} c_n p_n q_n f_n(x) - \sum_{n=1}^{\infty} c_n p_n q_n f_n(x) \\
&= 0 \\
&= (0) f(x)
\end{aligned}$$

$\left[\hat{P}, \hat{Q}\right] = 0$ for any function in Hilbert space, so $\hat{P}$ and $\hat{Q}$ commute. Therefore, two noncommuting operators cannot have a complete set of common eigenfunctions.